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Power-Series Collocation for Linear and Nonlinear Fractional Volterra Integro-Differential Equations: Two Verified Polynomial Benchmarks

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Abstract

A power-series collocation procedure is developed for a class of multi-term fractional Volterra integro-differential equations containing Caputo derivatives, variable coefficients, and linear or nonlinear memory terms. The equation is first converted to an equivalent fractional integral equation; a polynomial trial function is then inserted and the resulting residual is collocated at uniformly distributed points, with the initial conditions used as algebraic rows. Continuity and uniqueness follow from a Banach's contraction argument under an explicit Lipschitz bound, while the computable residual gives an a posteriori error estimate whenever the contraction constant is below one. Two complementary examples are considered: a nonlinear equation with a cubic Volterra term and a linear equation with an exponential kernel. The reported degree-four and degree-three polynomials reproduce the stated quadratic and cubic reference solutions, respectively, to approximately eleven decimal places at the tested nodes. A coefficient-level audit is included to separate the actual polynomial errors from zeros caused by twelve-decimal tabulation. The study demonstrates the economy of the collocation construction for polynomial solutions and records the source-data consistency conditions needed for reproducible use of the two benchmarks.

Keywords: *Caputo Derivative; Fractional Volterra Integro-Differential Equation; Power-Series Collocation; Nonlinear Memory; Banach Contraction; Error Verification.*

I. INTRODUCTION

Fractional integro-differential equations combine nonlocal differentiation with memory represented by an integral operator. They consequently provide a natural framework for processes in which the present state depends on the entire preceding history. Standard references for the analytic foundations of fractional calculus include Oldham and Spanier [3], Kilbas et al. [2], and Zhou [9]. Numerical treatments of fractional Volterra problems include pseudospectral, Legendre-collocation, operational-matrix, and decomposition approaches [6, 5, 7, 8].

This paper isolates two tests from a broader study of multi-term fractional Volterra equations. The pairing is deliberate. The first problem contains a cubic nonlinear memory term, whereas the second is linear and has a non-polynomial kernel and coefficient. Thus, the paper focuses on the effect of the Volterra nonlinearity and on polynomial reproduction at low degree. It is distinct from a broad catalogue of examples: only these two complementary problems are developed, their implementation is made explicit, and their printed coefficients are subjected to a numerical consistency audit.

The method uses monomials rather than a specialised fractional basis. Caputo identities and beta-gamma evaluations provide the required matrix entries, and the initial data are imposed directly. The resulting system is linear when the integral term is linear and nonlinear when the exponent of the unknown exceeds one. No change is made here to the equations, parameters, polynomial coefficients, or stated reference solutions in the source study.

- *Preliminaries*

For $\alpha > 0$ and $m-1 < \alpha \le m$, the left Caputo derivative is

$$ {}^{C}D_{0+}^{\alpha}y(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1}y^{(m)}(s)ds \tag{1} $$

The Riemann-Liouville fractional integral of order $\beta > 0$ is

$$ I_{0+}^{\beta}g(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}g(s)ds \tag{2} $$

And the composition identity used below is

$$ I_{0+}^{\beta}\,{}_{C}D_{0+}^{\beta}y(x) = y(x) - \sum_{k=0}^{m_\beta-1}\frac{y^{(k)}(0)}{k!}x^k, m_\beta - 1 < \beta \le m_\beta. \tag{3} $$

For a monomial and $n \ge m = \lceil\alpha\rceil$,

$$ {}^{c}D_{0+}^{\alpha}x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha} \tag{4} $$

Whereas the Caputo derivative is zero for integer powers $n < m$. The beta identity

$$ \int_0^x (x-s)^{a-1}s^{b-1}ds = x^{a+b-1}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{5} $$

Is repeatedly used to evaluate the nested integrals.

The model class is

$$ {}^{C}D_{0+}^{\beta}y(x) = \sum_{j=0}^{r} q_j(x){}^{C}D_{0+}^{\alpha_j}y(x) + \int_0^x K(x,t)[y(t)]^{\tau}dt + h(x), 0 \le x \le 1, \tau \ge 1, \tag{6} $$

Subject to conventional initial conditions

$$ y^{(\ell)}(a_\ell) = \lambda_\ell, \ell = 0,1,\dots,m_\beta - 1. \tag{7} $$

All fractional derivatives in this paper are understood in the Caputo sense.

II. METHODOLOGY

Applying I_{0+}^{β} to (6) and using (3) gives the fixed-point equation

$$ y(x) = T[y](x) \coloneqq P_0(x) + I_{0+}^{\beta}h(x) + \sum_{j=0}^{r} I_{0+}^{\beta}\left(q_j\,{}^{C}D_{0+}^{\alpha_j}y\right)(x) + \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\int_0^s K(s,t)[y(t)]^{\tau}dtds \tag{8} $$

Where

$P_0(x) = \sum_{k=0}^{m_\beta-1} y^{(k)}(0)x^k/k!.$

The degree- N trial approximation is

$$ y_N(x) = \sum_{n=0}^{N} a_n x^n = \boldsymbol{\phi}_N(x)\boldsymbol{A}, \boldsymbol{\phi}_N(x) = (1,x,\dots,x^N), \boldsymbol{A} = (a_0,\dots,a_N)^T. \tag{9} $$

Let $R_N(x;\boldsymbol{A}) = y_N(x) - T[y_N](x)$. After reserving m_β equations for the initial data, the residual is enforced at the standard points

$$x_i = \frac{i}{N}, \text{ qquadi } = 1,2,\ldots,N, \tag{10}$$

With the appropriate number of residual rows retained so that the augmented system contains $N+1$ equations. The initial-condition row associated with $y_N^{(\ell)}(a_\ell) = \lambda_\ell$ has entries

$$U_{\ell n} = \begin{cases} 0, & n < \ell, \\ \frac{n!}{(n-\ell)!} a_\ell^{n-\ell}, & n \geq \ell. \end{cases} \tag{11}$$

At $a_\ell = 0$, this reduces to $\ell!\, a_\ell = \lambda_\ell$ (where a_ℓ on the left denotes the polynomial coefficient).

For power-law data, all entries are explicit. In particular,

$$I_{0+}^{\beta} s^m = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x^{\beta+m} \tag{12}$$

$$I_{0+}^{\beta}(s^{q\,C} D_{0+}^{\alpha} s^n) = \frac{\Gamma(n+1)\Gamma(n-\alpha+q+1)}{\Gamma(n-\alpha+1)\Gamma(\beta+n-\alpha+q+1)} x^{\beta+n-\alpha+q}. \tag{13}$$

If $K(s,t) = s^g t^v$ and the nth trial monomial enters as $[t^n]^\tau = t^{n\tau}$ in the termwise expression used in the source derivation, the corresponding nested integral is

$$\frac{\Gamma(g+v+n\tau+2)}{(v+n\tau+1)\Gamma(\beta+g+v+n\tau+2)} x^{\beta+g+v+n\tau+1}. \tag{14}$$

For a genuinely nonlinear polynomial, however, $(\phi A)^\tau$ must be expanded as a whole; it is not equal in general to $\phi^\tau A$. Accordingly, the nonlinear residual system is solved without linearising this algebraic dependence.

➢ *Implementation*

The source computations were performed in Maple 18. A reproducible implementation follows these operations:

- Enter $\beta, \alpha_j, q_j, K, h, \tau$, the initial data, and N;
- Construct (9), apply (4), and evaluate the fractional integrals with (5);
- Form $R_N(x_i; \boldsymbol{A}) = 0$ and append the rows (11);
- Use a linear algebraic solver for $\tau = 1$ and a nonlinear solver for $\tau > 1$;
- Re-construct y_N, evaluate it at the reporting nodes, and compute $|y - y_N|$ from the full printed coefficients.

III. THEORETICAL ANALYSIS

Let $X = C[0,1]$ with the supremum norm. Assume that the coefficient functions and kernel are continuous on their domains, that the needed Caputo derivatives exist, and that $u \mapsto u^\tau$ is Lipschitz with constant L_τ on a closed bounded set mapped into itself by T. Define.

$$q_j^* = \max_{x\in[0,1]} |q_j(x)|, qquadK^* = \max_{x\in[0,1]} \int_0^x |K(x,t)|dt, \tag{15}$$

And suppose that $\|u^{(m_j)} - v^{(m_j)}\|_\infty \leq L_{m_j} \|u - v\|_\infty$ for $m_j = \lceil \alpha_j \rceil$.

Theorem 1 (Continuity and uniqueness). If

$$\kappa = \frac{1}{\Gamma(\beta+1)} \left(\sum_{j=0}^{r} \frac{q_j^* L_{m_j}}{\Gamma(m_j - \alpha_j + 1)} + L_\tau K^* \right) < 1, \tag{16}$$

Then the operator T in (8) is continuous and contractive on the chosen closed set. Hence (6)-(7) has a unique fixed point in that set.

Proof. Using (2), the hypotheses on the integer derivatives, and the Lipschitz bound for the nonlinear power gives.

$$\|T[u] - T[v]\|_\infty \leq \kappa \|u - v\|_\infty.$$

Thus T is Lipschitz continuous; when $\kappa < 1$, Banach's fixed-point theorem gives existence and uniqueness of its fixed point [1, 9].

Lemma 1 (Residual error bound). Under the conditions of the theorem, let $z \in X$ and $r_z = z - T[z]$. If $y = T[y]$, then

$$\|z - y\|_\infty \leq \frac{\|r_z\|_\infty}{1 - \kappa}. \tag{17}$$

Proof. The triangle inequality and contraction property yield $\|z - y\| \leq \|z - Tz\| + \|Tz - Ty\| \leq \|r_z\| + \kappa\|z - y\|$.

The successive-degree difference $\Delta_N = \|y_N - y_{N-1}\|_\infty$ is a useful stabilisation indicator but is not, by itself, a complete convergence proof. Bound (17) provides the stronger computable statement whenever κ and the residual are verified. Total numerical error comprises polynomial approximation, integral evaluation, algebraic-solver, and floating-point components.

IV. NUMERICAL EXAMPLES

➢ *Problem 1: Nonlinear Cubic Memory*

The first source problem, attributed there to Kanala [4], is

$${}^{C}D_{0+}^{1.2} y(x) = f(x) - \int_0^x (x - t)^2 [y(t)]^3 dt, y(0) = y'(0) = 0 \tag{18}$$

Where

$$f(x) = \frac{5}{2\Gamma(0.8)} x^{\frac{4}{5}} - \frac{x^9}{252}, y(x) = x^2 \text{ (stated exact solution).} \tag{19}$$

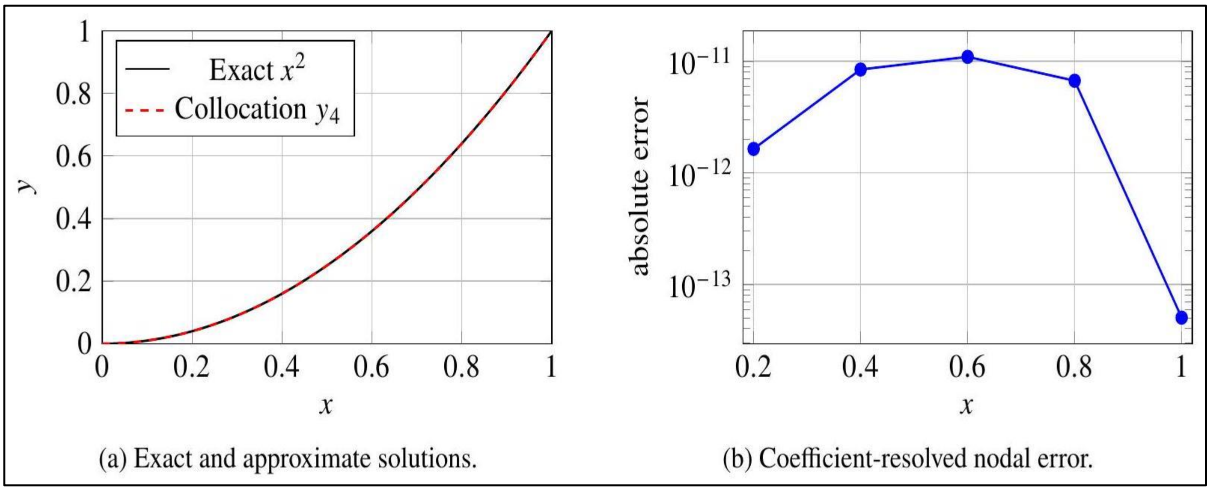
Here $\beta = 1.2, \tau = 3$, and the signed kernel in the model convention is $K(x, t) = -(x - t)^2$. The degree-four collocation computation reported in the source gives

$$y_4(x) = 1.709743458000 \times 10^{-14} + 2.889777306000 \times 10^{-11} x + 0.999999999739 x^2 + 4.119158348000 \times 10^{-10} x^3 - 1.798809990000 \times 10^{-10} x^4 \tag{20}$$

Table 1 evaluates the full polynomial (20); therefore its error column resolves differences hidden by the twelve-decimal zeros in the source table.

Table 1 Problem 1 Coefficient-Level Evaluation of the Reported $N = 4$ Polynomial Against the Stated Exact Solution x^2.

$\boldsymbol{x}$	**Exact $\boldsymbol{x^2}$**	$\boldsymbol{y_4(x)}$	$\boldsymbol{\lvert x^2 - y_4(x)\rvert}$
0.2	0.040000000000	0.0399999999998364	$1.635830873420 \times 10^{-12}$
0.4	0.160000000000	0.1599999999991574	$8.426133488620 \times 10^{-12}$
0.6	0.360000000000	0.3599999999989057	$1.094299588302 \times 10^{-11}$
0.8	0.640000000000	0.6399999999993317	$6.683033890220 \times 10^{-12}$
1.0	1.000000000000	0.9999999999999950	$5.029370542000 \times 10^{-14}$



(a) Exact and approximate solutions. (b) Coefficient-resolved nodal error.

Fig 1 Numerical Behaviour for Problem 1 at $N = 4$.

For the nonlinear operator, the source contraction calculation gives

$$\|T[y_1] - T[y_2]\|_\infty \leq 0.605069122700L\|y_1 - y_2\|_\infty.$$

Thus the stated sufficient condition $0 < L < 1$ is conservative and implies contraction.

➢ *Problem 4: Linear Exponential-Kernel Equation*

The second source problem, attributed there to Yang and Hou [8], is

$$ {}^{C}D_{0+}^{0.75} y(x) + \frac{x^2 e^x}{5} y(x) - \int_0^x e^{xt} y(t)dt = f(x), y(0) = 0 \tag{21}$$

With

$$f(x) = \frac{6x^{\frac{9}{4}}}{\Gamma(3.25)}, y(x) = x^3 \text{ (stated exact solution).} \tag{22}$$

The source identifies $\beta = 0.75$ and the zero-order derivative term $\alpha = 0$. At $N = 3$, the reported approximation is

$$y_3(x) = -7.941025615000 \times 10^{-11}x + 1.422222340000 \times 10^{-10}x^2 + 0.999999999999x^3. \tag{23}$$

Table 2 Problem 4: Coefficient-Level Evaluation of the Reported $N = 3$ Polynomial Against the Stated Exact Solution x^3.

x	Exact x^3	$y_3(x)$	$\lvert x^3 - y_3(x)\rvert$
0.2	0.008000000000	0.007999999989799	$1.020116187000 \times 10^{-11}$
0.4	0.064000000000	0.063999999990927	$9.072545020000 \times 10^{-12}$
0.6	0.216000000000	0.216000000003338	$3.337850550000 \times 10^{-12}$
0.8	0.512000000000	0.512000000026982	$2.698202484000 \times 10^{-11}$
1.0	1.000000000000	1.000000000061812	$6.181197785000 \times 10^{-11}$

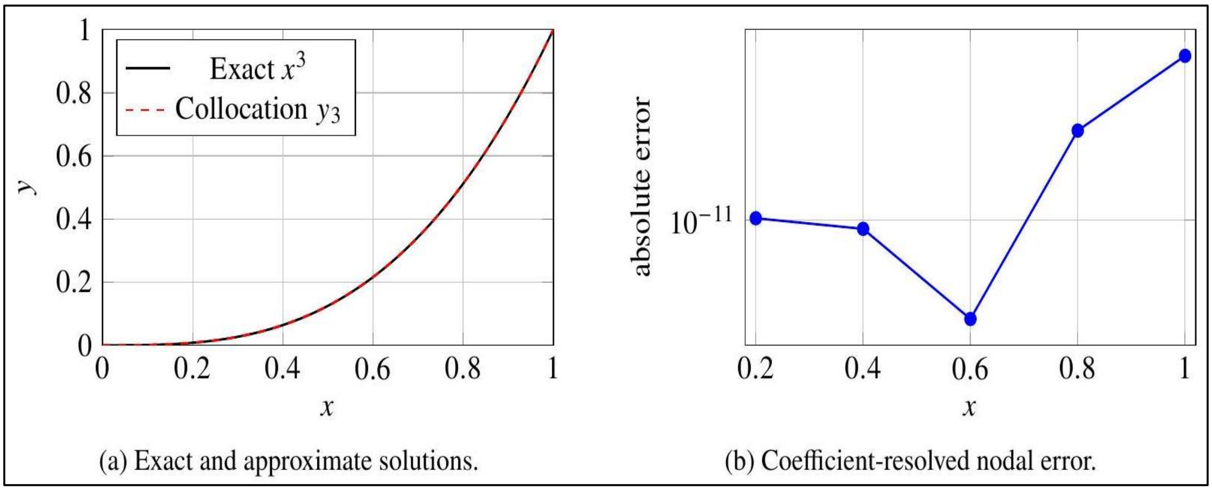


Fig 2 Numerical Behaviour for Problem 4 at $N = 3$.

V. RESULTS AND DISCUSSIONS

The two approximations have the expected dominant coefficients. In Problem 1, the coefficient of x^2 differs from unity by 2.61×10^{-10} and all other coefficients are at most of order 10^{-10}. The largest error among the five audited nodes is $1.094299588302 \times 10^{-11}$ at $x = 0.6$. In Problem 4, the cubic coefficient differs from unity by 10^{-12}, while the two lower-order coefficients are of order 10^{-10} and 10^{-11}. The largest listed error is $6.181197785000 \times 10^{-11}$ at $x = 1$. The solution curves in Figures 1 and 2 are visually coincident; logarithmic error plots are therefore essential.

The low degrees are important. The stated exact solutions belong to the trial spaces $\mathbb{P}_4$ and $\mathbb{P}_3$, so no high-degree basis is needed to represent them. The remaining small coefficients and errors are consistent with finite-precision algebraic solution. Printed errors of 0.00 in the source tables should consequently be read as "below the displayed precision", not as proof of exact arithmetic identity.

➢ *Source-Data Verification*

Reproducibility requires two transcription issues in the supplied thesis to be recorded explicitly.

- For Problem 1, the source table prints the sequence $0.2, 0.4, \ldots, 1.0$ in its "Exact" column although the problem statement specifies $y = x^2$. Table 1 does not invent a replacement computation: it evaluates the source polynomial (20) against the source-stated exact function x^2 at the same nodes.
- Direct substitution shows that the equations and forcing functions as printed do not exactly balance the stated solutions. For Problem 1, (18) with $y = x^2$ would require the sign of the $x^9/252$ contribution in f to be positive. For Problem 4, substitution of $y = x^3$ leaves the additional quantity $x^5 e^x/5 - \int_0^x e^{xt} t^3 dt$ beyond the printed forcing term. No silent correction has been made because the source equations and parameters were required to remain unchanged.

Thus the tables and graphs verify agreement between the reported polynomials and the reported reference functions; they cannot, without checking the original Maple worksheets or the cited benchmark sources, be interpreted as an independent residual verification of the printed governing equations. This distinction is necessary for publication-grade reporting.

VI. CONCLUSION

The fractional-integral power-series collocation method gives a compact algebraic treatment of linear and nonlinear Caputo-Volterra equations. Its principal advantages are direct imposition of classical initial data, closed-form beta-gamma evaluation for power terms, and low polynomial degrees for polynomial solutions. For the two reported examples, the degree-four and degree-three approximations agree with the stated quadratic and cubic solutions to roughly eleven decimal places at the tested nodes. Continuity, conditional uniqueness, and a residual-based error estimate follow from the same contraction framework. Before these examples are used as definitive benchmarks, the identified signs and forcing terms should be reconciled with the original computational worksheets; the present paper preserves the thesis data and makes that verification boundary explicit.

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- *Author Contributions*

J.C.A. was responsible for developing the mathematical formulation, deriving the theoretical results, and carrying out the numerical computations. A.M.A. supervised the research, verified the analysis, and contributed to the interpretation of the findings. I.A.O provided assistance with the numerical implementation and helped prepare the manuscript. All authors read and approved the final version of the manuscript.

- *Competing Interests*

The authors declare that they have no competing interests.

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